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Nonlinear instability theories in hydrodynamics

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Abstract

After over one hundred years of effort in theory, experiment and calculation, linear and nonlinear approaches to instability processes remain clouded in controversy. In this review some of the existing theories will be summarized.

Given the Navier–Stokes equations for an incompressible fluid

$$\boldsymbol{N}(\boldsymbol{u}) = \boldsymbol{0},$$

and a basic steady solution

$$\boldsymbol{u} = \bar{\boldsymbol{u}}(\boldsymbol{x}),$$

the linear theory yields a set of partial differential equations, namely

$$L(\boldsymbol{u}'(\boldsymbol{x}, t)) = 0.$$

For modal solutions of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{x}+\sigma t}$ there is an eigenvalue problem for σ. If $Re(\sigma) > 0$ the flow is linearly unstable.

Such a problem is a common and classical one in physical applied mathematics. An original hope was to use linear instability to find a route from laminar to turbulent flow. While there have been some partial successes there have also been many frustrations and difficulties.

In this talk, we plan to concentrate on homogeneous shear flows so that only one dimensionless parameter, the Reynolds number $R = Ul/v$ is involved. The equations are

$$\operatorname{div} \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \frac{1}{R}\triangle \boldsymbol{u},$$

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satisfied by the steady quasi-parallel flow solution $\overline{\boldsymbol{u}} = (\bar{u}(y), 0, 0)$. Linear theory then yields

$$\operatorname{div} \boldsymbol{u}' = 0,$$

$$\frac{\partial \boldsymbol{u}'}{\partial t} + \bar{u}(y)\boldsymbol{i} \cdot \nabla \boldsymbol{u}' + \boldsymbol{u}' \cdot \nabla \bar{u}\boldsymbol{i} = -\nabla p' + \frac{1}{R} \triangle \boldsymbol{u}',$$

and for normal mode solutions of the form

$$f' = \hat{f}(y)\mathrm{e}^{\mathrm{i}\alpha x + \mathrm{i}\beta z - \mathrm{i}\alpha c t}$$

the relevant linear equations are

$$\mathrm{i}\alpha\hat{u} + \frac{\mathrm{d}\hat{v}}{\mathrm{d}y} + \mathrm{i}\beta\hat{w} = 0,$$

$$\mathrm{i}\alpha(\bar{u} - c)\hat{u} + \frac{\mathrm{d}\bar{u}}{\mathrm{d}y}\hat{v} = -\mathrm{i}\alpha\hat{p} + \frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2 - \beta^2\right)\hat{u},$$

$$\mathrm{i}\alpha(\bar{u} - c)\hat{v} = \frac{-\mathrm{d}\hat{p}}{\mathrm{d}y} + \frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2 - \beta^2\right)\hat{v},$$

$$\mathrm{i}\alpha(\bar{u} - c)\hat{w} = -\mathrm{i}\beta\hat{p} + \frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2 - \beta^2\right)\hat{w}.$$

This system is readily reduced to two equations for the vertical velocity $\hat{v}(y)$ and the vertical vorticity $\hat{\eta} = \mathrm{i}\beta\hat{u} - \mathrm{i}\alpha\hat{w}$, namely

$$\frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2 - \beta^2\right)^2\hat{v} - \mathrm{i}\alpha\left[(\bar{u} - c)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2 - \beta^2\right)\hat{v} - \bar{u}_{yy}\hat{v}\right] = 0,$$

$$\frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \alpha^2 - \beta^2\right)\hat{\eta} - \mathrm{i}\alpha(\bar{u} - c)\hat{\eta} = \mathrm{i}\beta\hat{u}_y\hat{v}.$$

Typical boundary conditions are $\hat{v} = \mathrm{d}\hat{v}/\mathrm{d}y = \hat{\eta} = 0$ at $y = y_1, y_2$.
(These correspond to rigid boundaries at $y = y_1, y_2$).
Historically the main focus has been on the Orr–Sommerfeld equation (the fourth-order equation for $\hat{v}$). This equation is equivalent to that for a two dimensional disturbance via

$$\alpha^2 + \beta^2 = \bar{\alpha}^2,$$

$$c = \bar{c},$$

$$\alpha R = \bar{\alpha}\bar{R},$$

this result, due to Squire [6], resulted in research concentrating on two-dimensional disturbances.
Attention will now be confined to summarizing existing nonlinear developments.
In the early 1960s, Stuart [7] presented a two-dimensional theory that relaxes the exponential temporal growth in order to account for nonlinear effects. The expansion for v proceeds

as follows:

$$\begin{aligned} v = {} & A(t)\hat{v}_1(y)\mathrm{e}^{\mathrm{i}\alpha x} + A^*(t)\hat{v}_1^*(y)\mathrm{e}^{-\mathrm{i}\alpha x} \\ & + AA^* v_0(y) + A^2\hat{v}_2(y)\mathrm{e}^{2\mathrm{i}\alpha x} + A^{*2}\hat{v}_2^*(y)\mathrm{e}^{-2\mathrm{i}\alpha x} \\ & + A^2A^* v_{11}(y)\mathrm{e}^{\mathrm{i}\alpha x} + A^{*2}A v_{11}^*(y)\mathrm{e}^{-\mathrm{i}\alpha x} \\ & + A^3\hat{v}_3(y)\mathrm{e}^{3\mathrm{i}\alpha x} + A^{*3}A\hat{v}_3^*(y)\mathrm{e}^{-3\mathrm{i}\alpha x} + \cdots . \end{aligned}$$

Analysis shows that the leading order nonlinear effect gives the evolution equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\mathrm{i}\alpha c A + KA^2A^* + \cdots .$$

Here K is the (complex) Landau constant. This procedure can be generalized to include several modes and spatial dependence. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}|A|^2 = 2\alpha c_i|A|^2 + K_r|A|^4 + \cdots$$

which leads to the existence of potential finite amplitude equilibrium states.

This theory does not lead to a satisfactory explanation for many experimental observations. For example, experiments show strong three-dimensional effects especially with respect to mean flows. Local bursts of high frequency arise signifying the first onset of turbulence. Furthermore, the theoretical expansion is not well ordered for large Reynolds numbers (R) unless the amplitude ε is extremely small. The relevant parameter is found to be $\varepsilon^{1/2}R^{1/3}$. Such a theory has been explored by Maslowe [5] and Haberman [4], i.e., when $\varepsilon^{1/2}R^{1/3} \gg 1$, but does have limitations.

In reality for three-dimensional disturbances there are two degrees of freedom so that if $A(t)$ and $B(t)$ are associated with the amplitudes $\hat{v}$ and $\hat{\eta}$ it is a relatively simple calculation to show that A and B satisfy equations of the form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\mathrm{i}\alpha c_1 A + KA^2A^*,$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -\mathrm{i}\alpha c_2 B + LAA^*B.$$

Another possibility has been explored by Gustavsson [3] is that $c_1 \approx c_2$, a so-called double resonance. In such solutions nonlinear effects develop rapidly but since the vertical vorticity mode is always damped, at least slightly, the possibilities remain restricted.

Triad resonances have been considered by Craik [2] and Zhou [8]. This is a standard mechanism for energy transfer among dispersive waves, but the process is less clear when linear instability mechanisms are involved.

A stronger nonlinear theory has been proposed by Benney and Chow [1]. This so-called mean flow first harmonic theory will now be outlined.

For a single mode consider an expansion of the form

$$u = u_0 + \varepsilon(u_1\mathrm{e}^{\mathrm{i}\theta} + *) + \varepsilon^2(u_2\mathrm{e}^{2\mathrm{i}\theta} + *) + \cdots$$

$$v = \varepsilon v_0 + \varepsilon(v_1\mathrm{e}^{\mathrm{i}\theta} + *) + \varepsilon^2(v_2\mathrm{e}^{2\mathrm{i}\theta} + *) + \cdots$$

$$w = \varepsilon w_0 + \varepsilon(w_1 e^{i\theta} + *) + \varepsilon^2(w_2 e^{2i\theta} + *) + \cdots$$

$$p = \varepsilon^2 p_0 + \varepsilon(p_1 e^{i\theta} + *) + \varepsilon^2(p_2 e^{2i\theta} + *) + \cdots$$

where $\theta = \frac{1}{\varepsilon}\Theta(X,T), \quad \alpha = \Theta_X,$

$$(X,T) = \varepsilon(x,t),$$

$$f_n = f_n(X,y,z,T).$$

In the inviscid case, the leading order equations are:

$$\frac{\partial}{\partial y}\left(\frac{\partial p_1/\partial y}{(u_0-c)^2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial p_1/\partial z}{(u_0-c)^2}\right) - \frac{\alpha^2 p_1}{(u_0-c)^2} = 0,$$

$$\frac{\partial u_0}{\partial X} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0,$$

$$\frac{\partial u_0}{\partial T} + u_0\frac{\partial u_0}{\partial X} + v_0\frac{\partial u_0}{\partial y} + w_0\frac{\partial u_0}{\partial z} = 0,$$

$$\frac{\partial v_0}{\partial T} + u_0\frac{\partial v_0}{\partial X} + v_0\frac{\partial v_0}{\partial y} + w_0\frac{\partial v_0}{\partial z} + \frac{2}{\alpha^2}\frac{\partial}{\partial y}(S_{22}) + \frac{1}{\alpha^2}\frac{\partial}{\partial z}(S_{23}) = \frac{-\partial p_0}{\partial y},$$

$$\frac{\partial w_0}{\partial T} + u_0\frac{\partial w_0}{\partial X} + v_0\frac{\partial w_0}{\partial y} + w_0\frac{\partial w_0}{\partial z} + \frac{1}{\alpha^2}\frac{\partial}{\partial y}(S_{23}) + \frac{2}{\alpha^2}\frac{\partial}{\partial z}(S_{33}) = \frac{-\partial p_0}{\partial z},$$

where

$$S_{22} = \frac{|\partial p_1/\partial y|^2}{(u_0-c)^2}, \quad S_{33} = \frac{|\partial p_1/\partial z|^2}{(u_0-c)^2},$$

$$S_{23} = \frac{\partial p_1/\partial y\, \partial p_1^*/\partial z + \partial p_1^*/\partial y\, \partial p_1/\partial z}{(u_0-c)^2}.$$

One exact solution to the above equations is given by

$$u_0 = \bar{u}_0(y), \quad v_0 = w_0 = 0,$$

$$p_1 = \bar{p}_1(y), \quad c = \bar{c}$$

in which case the eigenvalues $\bar{c}$ are solutions of

$$\frac{d}{dy}\left(\frac{d\bar{p}_1/dy}{(\bar{u}_0-\bar{c})^2} - \frac{\alpha^2\bar{p}_1}{(\bar{u}_0-\bar{c})^2}\right) = 0,$$

$$\text{with } \frac{d\bar{p}_1}{dy} = 0, \ \ y = 0; \quad \frac{d\bar{p}_1}{dy} - \alpha^2(\bar{u}_0-\bar{c})^2\bar{p}_1 = 0, \ \ y = 1.$$

We are now in a position to examine the second stability problem, namely that of shear flow plus the wave. For this purpose, a second linearization is considered, namely,

$$p_1 = \bar{p}_1(y) + \varepsilon' p_1(y)\mathrm{e}^{\mathrm{i}\beta z + \mathrm{i}K(X-CT)},$$

$$v_0 = \varepsilon' v_0(y)\mathrm{e}^{\mathrm{i}\beta z + \mathrm{i}K(X-CT)}.$$

A fourth-order eigenvalue problem results to determine the new eigenvalues $\bar{C}$:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\mathrm{d}p_1/\mathrm{d}y}{(\bar{u}_0-\bar{c})^2}\right) - \frac{\alpha^2+\beta^2}{(\bar{u}_0-\bar{c})^2}p_1 - \frac{2d\bar{p}_1/\mathrm{d}y}{(\bar{u}_0-\bar{c})^2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{(\mathrm{d}\bar{u}_0/\mathrm{d}y)v_0}{(\bar{u}_0-\bar{c})(\bar{u}_0-C)}\right) = 0,$$

$$(\bar{u}_0 - C)\left(\frac{\mathrm{d}^2 v_0}{\mathrm{d}y^2} - \beta^2 v_0\right) - \frac{\mathrm{d}^2\bar{u}_0}{\mathrm{d}y^2}v_0 + \frac{8\beta^2}{K^2}\frac{(\mathrm{d}\bar{u}_0/\mathrm{d}y)v_0}{(\bar{u}_0-C)}\left(\frac{\bar{p}_1\mathrm{d}\bar{p}_1/\mathrm{d}y}{(\bar{u}_0-\bar{c})^3} + \frac{\mathrm{d}\bar{u}_0/\mathrm{d}y(\mathrm{d}\bar{p}_1/\mathrm{d}y)^2}{\alpha^2(\bar{u}_0-\bar{c})^4}\right),$$

$$-\frac{4\beta^2}{K^2}\frac{\mathrm{d}\bar{u}_0/\mathrm{d}y}{(\bar{u}_0-\bar{c})}\left(\bar{p}_1 p_1 + \frac{1}{\alpha^2}\frac{\mathrm{d}\bar{p}_1}{\mathrm{d}y}\frac{\mathrm{d}p_1}{\mathrm{d}y}\right) = 0,$$

$$\text{with } v_0 = \frac{\mathrm{d}p_1}{\mathrm{d}y} = 0, \quad y = 0,$$

$$v_0 = \frac{\mathrm{d}p_1}{\mathrm{d}y} - \alpha^2(\bar{u}_0 - c)^2 p_1 = 0, \quad y = 1.$$

Numerical and analytical calculations show that Im(KC) > 0, in general, so that linearly unstable solutions exists. These solutions can be generalized to three-dimensional modes, many disturbances and viscous fluids.

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